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Exponential convergence in entropy of the McKean-Vlasov equation

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1. McKean-Vlasov equation

consider the non-linear McKean-Vlasov equation with

• a confinement potential $V : \mathbb{R}^d \to \mathbb{R}$ and

• an interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (between two particles) so that $W(x,y) = W(y,x)$:

$$\partial_t \nu_t = \Delta \nu_t + \nabla \cdot (\nu_t \nabla V) + \nabla \cdot (\nu_t \nabla (W \circledast \nu_t)) \tag{1}$$

where $(\nu_t)_{t \geq 0}$ is a flow of probability measures on $\mathbb{R}^d$ with ν_0 given, ∇ is the gradient, $\nabla\cdot$ is the divergence, and

$$(W \circledast \nu)(x) = \int_{\mathbb{R}^d} W(x,y) d\nu(y). \tag{2}$$

It corresponds to the self-interacting diffusion

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)dt - \nabla W \circledast \nu_t(X_t)dt \qquad (3)$$

where ν_t is the law of X_t.

An important question is

1) whether does $\lim_{t\to+\infty}\nu_t = \nu_\infty$ exist ?

The basic assumption is that ν_∞, equilibrium state of the MV equation

$$\Delta\nu_\infty + \nabla\cdot(\nu_\infty\nabla V) + \nabla\cdot(\nu_\infty\nabla(W \circledast \nu_\infty)) = 0$$

must be unique. In statistical physics, that means no phase transition.

2) Is there the exponential convergence in the above convergence ?

Carrillo-McCann-Villani's theorem

Introduce the free energy

$$\begin{aligned} E_f(\nu) &:= H(\nu|\alpha) + \frac{1}{2}\iint W(x,y)d\nu(x)d\nu(y) \\ &= H(\nu|\exp(-V(x) - \frac{1}{2}W \circledast \nu)) + c \end{aligned} \tag{4}$$

where

$$\alpha(dx) = \frac{1}{C}e^{-V(x)}dx.$$

Then solution ν_∞ of the stable MV equation is the critical point of E_f.

The corresponding mean field entropy

$$H_W(\nu) := E_f(\nu) - \inf_{\nu \in \mathcal{M}_1(\mathbb{R}^d)} E_f(\nu). \tag{5}$$

Theorem 1 **(Carrillo-McCann-Villani 03)** *Assume that*

$$\nabla^2 V \geq \gamma I, \gamma > 0$$

and $W(x, y) = W_0(x - y)$ *with even and convex* W_0. *Then*

$$H_W(\nu_t) \leq e^{-\gamma t} H_W(\nu_0).$$

Ideas:

1) The MV equation is the gradient flow

$$\partial_t \nu_t = -\nabla_{OV} H_W(\nu_t)$$

2) Prove that

$$\nabla^2_{OV} H_W \geq \gamma.$$

Historical comments:

Carrillo-McCann-Villani 03

Malriau 03

Cattiaux-Guillin-Malriau 08

Bolley-Gentil-Guillin 13

Cattiaux-Guillin-Zhang 19

A. Durmus, A. Eberle, A. Guillin, and R. Zimmer 19 (the first work in the double well case).

F.Y. Wang 21

2. Mean-field particle system

The McKean-Vlasov equation is the ideal limiting model of mean field particle system below

$$\begin{aligned} dX_i^N(t) &= \sqrt{2}dB_i(t) - \nabla V(X_i^N(t))dt \\ &\quad - \frac{1}{N-1}\sum_{j\neq i} \nabla_x W(X_i^N(t), X_j^N(t))dt, \end{aligned} \tag{6}$$

where $i = 1, \cdots, N, B_1(t), \cdots, B_N(t)$ are N independent Brownian motions taking values in $\mathbb{R}^d$.

Its generator $\mathcal{L}^{(N)}$ is given by

$$\mathcal{L}^{(N)} f(x_1, \cdots, x_N) = \sum_{i=1}^{N} \mathcal{L}_i^{(N)} f(x_1, \cdots, x_N)$$
$$\mathcal{L}_i^{(N)} f := \Delta_i f - \nabla_i V(x_i) \cdot \nabla_i - \frac{1}{N-1} \sum_{j \neq i} (\nabla_x W)(x_i, x_j) \cdot \nabla_i f \tag{7}$$

for any smooth function f on $(\mathbb{R}^d)^N$, where ∇_i denotes the gradient w.r.t. x_i, Δ_i the Laplacian w.r.t. x_i, and $x \cdot y = \langle x, y \rangle$ denotes the Euclidean inner product.

The unique invariant probability measure of (6) is

$$\mu^{(N)} = \frac{1}{Z_N} \exp\{-H_N(dx_1, \cdots, dx_N)\} dx_1 \cdots dx_N \tag{8}$$

where

$$H_N(x_1, \cdots, x_N) := \sum_{i=1}^{N} V(x_i) + \frac{1}{N-1} \sum_{1 \le i < j \le N} W(x_i, x_j)$$

is the Hamiltonian, and Z_N is the normalization constant called *partition function* in statistical mechanics, which is assumed to be finite throughout the paper. Without interaction (i.e. $W = 0$ or constant), $\mu^{(N)} = \alpha^{\otimes N}$ (i.e. the particles are independent), where

$$d\alpha(x) = \frac{1}{C} e^{-V(x)} dx, \ C = \int e^{-V(x)} dx.$$

Formally

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i} \to \nu_t$$

and then

$$\text{law of } (X_t^1) \to \nu_t.$$

That is the theory of propagation of chaos, see Sznitmann 91.

How to establish propagation of chaos when W is singular is a very challenging question: much recent progresses!

Our goal:

to remove the convexity assumption in the work of Carrillo-McCann-Villani.

Our approach:

To establish the uniform log-Sobolev inequality for $\boldsymbol{\mu}^{(N)}$ and show that it implies the exponential convergence of the MV equation.

3. Uniform Poincaré inequality

Introduce the dissipativity rate of the drift of one particle at distance $r > 0$:

$$b_0(r) \geq - \langle \frac{x-y}{|x-y|}, (\nabla V(x) - \nabla V(y)) + (\nabla_x W(x,z) - \nabla_x W(y,z)) \rangle$$

for all $x, y, z \in \mathbb{R}^d : |x-y| = r$.

This dissipativity coefficient appeared in Chen and Wang (95, 96, 97).

Assume the **Lipschitzian spectral gap condition for one particle**

$$c_{Lip,m} := \frac{1}{4}\int_0^\infty \exp\left\{\frac{1}{4}\int_0^s b_0(u)\mathrm{d}u\right\} s\mathrm{d}s < +\infty. \qquad (9)$$

In fact in W. (JFA09), we have proved that for one single particle generator

$$\|[\Delta - (\nabla U - \nabla_x W(\cdot, z)\cdot\nabla)]^{-1}\|_{Lip} \le c_{Lip,m}.$$

Theorem 2 *(Essentially due to Ledoux 01) Assume that there is some constant $h > -\frac{1}{c_{Lip,m}}$ such that for any $(x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$,*

$$\frac{1}{N-1}(1_{i\neq j}\nabla^2_{x,y}W(x_i, x_j))_{1\leq i,j\leq N} \geq hI_{dN} \tag{10}$$

in the order of definite nonnegativity for symmetric matrices, where I_n is the identity matrix of order n. Then $m = m^{(N)}$ satisfies the following uniform Poincaré inequality

$$\left(\frac{1}{c_{Lip,m}} + h\right) \mathrm{Var}_{\mu^{(N)}}(f) \leq \int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)} \tag{11}$$

One can also use the result of W. (AP06) to obtain a similar result based on the Dobrushin's uniqueness condition, which becomes different from (10).

4. Uniform log-Sobolev inequality

Theorem 3 *(Guillin, Liu, Wu, Zhang 19) Assume that*

1. *for some best constant $\rho_{LS,m} > 0$, the conditional marginal distributions $\mu_i := \mu(dx_i|x^{\hat{i}})$ on $\mathbb{R}^d$ satisfy the log-Sobolev inequality :*

$$\rho_{LS,m} Ent_{\mu_i}(f^2) \leq 2 \int |\nabla f|^2 d\mu_i$$

for all i and $x^{\hat{i}}$;

2. $$\gamma_0 = c_{Lip,m} \sup_{x,y \in \mathbb{R}^d, |z|=1} |\nabla^2_{x,y} W(x,y) z| < 1. \tag{12}$$

which implies Dobrushin-Zegarlinski uniqueness condition (no phase transition) *then* $\mu^{(N)}$ *satisfies the uniform log-Sobolev inequality*

$$\frac{\rho_{\mathrm{LS,m}}}{(1-\gamma_0)^2} Ent_{\mu^{(N)}}(f^2) \le 2 \int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)}$$

Or equivalently

$$\frac{\rho_{\mathrm{LS,m}}}{(1-\gamma_0)^2} H(\nu|\mu^{(N)}) \le 2 I(\nu|\mu^{(N)})$$

where for $\nu = h\mu^{(N)}$

$$I(\nu|\mu^{(N)}) = \int |\nabla \sqrt{h}|^2 d\mu^{(N)} = \frac{1}{4} \int \frac{|\nabla h|^2}{h} d\mu^{(N)},$$

is the Fisher-Donsker-Varadhan information.

The difficulty:

How to verify the condition of Dobrushin-Zegarlinski ?

The key :

the result of the Lipschitzian norm of $(-\mathcal{L}_1)^{-1}$ for one single particle, obtained by W. (JFA09).

5. Exponential convergence in entropy of MV

The substituter of the Fisher-Donsker-Varadhan's information for the MV equation is: if $\nu = f(x)dx, \int |x|^2 d\nu(x) < +\infty$ and $\nabla f \in L^1_{loc}(\mathbb{R}^d)$ in the distribution sense,

$$I_W(\nu) := \frac{1}{4} \int |\frac{\nabla f(x)}{f(x)} + \nabla V(x) + (\nabla_x W \circledast \nu)(x)|^2 d\nu(x), \tag{13}$$

and $+\infty$ otherwise.

Theorem 4 *Assume the uniform marginal log-Sobolev inequality, i.e. $\rho_{LS,m} > 0$, and the uniqueness condition (12). Then*

1. *the minimizer ν_∞ of H_W over $\mathcal{M}_1(\mathbb{R}^d)$ is unique;*

2. *the following (nonlinear) log-Sobolev inequality*

$$\rho_{LS} H_W(\nu) \le 2 I_W(\nu), \; \nu \in \mathcal{M}_1(\mathbb{R}^d) \tag{14}$$

holds, where

$$\rho_{LS} := \limsup_{N \to \infty} \rho_{LS}(\mu^{(N)}) \ge \frac{\rho_{LS,m}}{(1-\gamma_0)^2}.$$

3. *The following Talagrand's transportation inequality holds*

$$\rho_{LS} W_2^2(\nu, \nu_\infty) \le 2 H_W(\nu), \; \nu \in \mathcal{M}_1(\mathbb{R}^d) \tag{15}$$

4. *for the solution ν_t of the McKean-Vlasov equation with the given initial distribution ν_0 of finite second moment,*

$$H_W(\nu_t) \le e^{-t \cdot \rho_{LS}/2} H_W(\nu_0), \; t \ge 0 \tag{16}$$

and in particular

$$W_2^2(\nu_t, \nu_\infty) \leq \frac{2}{\rho_{LS}} e^{-t\cdot\rho_{LS}/2} H_W(\nu_0),\ t \geq 0 \qquad (17)$$

Ideas of proof :

$$\frac{1}{N} H(\nu^{\otimes N} | \mu^{(N)}) \to H_W(\nu)$$

and

$$\frac{1}{N} I(\nu^{\otimes N} | \mu^{(N)}) \to I_W(\nu)$$

6. Two examples

Example 1 (**Curie-Weiss model**) *Let* $d = 1$, $V(x) = \beta(x^4/4 - x^2/2)$, $W(x, y) = -\beta K xy$ *where* $\beta > 0$. *We have*

$$c_{Lip,m} \leq \sqrt{\frac{\pi}{\beta}} e^{\beta/4}.$$

So that

$$\gamma_0 \leq c_{Lip,m} \|\nabla^2_{x,y} W\|_\infty \leq \sqrt{\pi\beta} e^{\beta/4} |K|$$

which will be smaller than 1 if β *or* K *is sufficiently small.*

Example 2 **(Quadratic interaction model)** *Let* $d = 1$, $V(x) = \beta(x^4/4 - x^2/2)$, $W(x,y) = W_0(x-y)$, $W_0(z) = -\beta K z^2/2$ *where* $\beta, K > 0$. *We have*

$$c_{Lip,m} \leq \sqrt{\pi/\beta}\, e^{\beta(1+K)^2/4}$$

and

$$\gamma_0 \leq \sqrt{\pi\beta}\, K e^{\beta(1+K)^2/4}.$$

Thanks !